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$E_{7(7)}$ on the light cone

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Abstract: We use the Cremmer-Julia $E_{7(7)}$ non-linear symmetry of $\mathcal{N} = 8$ Supergravity to derive its order κ^2 on-shell Hamiltonian in terms of one chiral light-cone superfield. By requiring that $E_{7(7)}$ commute with the super-Poincaré group, we deduce to lowest non-trivial order in κ, the light cone $E_{7(7)}$ transformations of all fields of the theory, including the graviton. We then derive the dynamical supersymmetry transformation to order κ^2, and express the Hamiltonian as a quadratic form in the chiral superfield.

Keywords: Extended Supersymmetry, Superspaces, Supergravity Models.

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Contents

1. Introduction

The maximally supersymmetric $\mathcal{N} = 8$ Supergravity [1] and $\mathcal{N} = 4$ SuperYang-Mills [2] play a very important rôle in modern theory. In the standard descriptions they look quite different and are naturally related to eleven- and ten-dimensional theories, respectively. In the light-cone frame description, however, they are described in a remarkably similar way hinting at a deep relation between them. In four dimension, they are the only two (except possibly for some higher-spin theories) that are described by one chiral *constrained* light-cone superfield which captures *all* their physical degrees of freedom [3]. Also, tree level Supergravity amplitudes are related to the square of Yang-Mills amplitudes [4, 5], and the light-cone Hamiltonian of both theories can be written as a *positive definite* quadratic form in their superfields [6, 7]. Some even suggest that the ultraviolet finiteness of $\mathcal{N} = 4$ SuperYang-Mills [8] might extend to $\mathcal{N} = 8$ Supergravity [9, 10]. There are important structural differences though. $\mathcal{N} = 8$ Supergravity, unlike $\mathcal{N} = 4$ SuperYang-Mills, is not Superconformal invariant. Instead it has the on-shell, non-linear Cremmer-Julia, $E_{7(7)}$ duality symmetry [1]. It is therefore natural to ask if this symmetry can be exploited to bring simplicty to the quartic and higher-order interactions of $\mathcal{N} = 8$ Supergravity. In this letter, as a first step in this direction, we show how to exploit this symmetry to construct the light-cone Hamiltonian to order κ^2. Our resulting expression is remarkably simpler than a recent formulation of the same Hamiltonian with over ninety terms [7]. In this process we will also get the $E_{7(7)}$ transformations to lowest order for all the fields in the theory. The details of the calculations will be presented elsewhere [11].

After a brief review of $E_{7(7)}$ duality in the covariant formalism with the scalar and field strengths alone, we express the action of $E_{7(7)}$ in the LC_2 formalism where all unphysical degrees of freedom have been eliminated. The explicit non-linear $E_{7(7)}$ action on the scalars and vector potentials to lowest non-trivial order in κ are derived in this gauge; they stand as the starting point for our analysis.

The remaining fields of $\mathcal{N}=8$ Supergravity, including the graviton, alter these transformations through Supersymmetry. The kinematical supersymmetries generate linear transformations on the chiral superfield, while the dynamical supersymmetries are non-linear. The requirement that $E_{7(7)}$ commute with the kinematical light-cone Supersymmetries yields the $E_{7(7)}$ transformations of all fields, including the graviton. The order-κ $E_{7(7)}$ transformations can then be expressed in terms of transformations on the superfield.

Extending the commutativity to the dynamical Supersymmetries enables us then to derive through algebraic consistency, the order-κ^2 form of the supersymmetry transformations. The quartic light-cone Hamiltonian follows.

2. Covariant $\mathcal{N}=8$ supergravity

$\mathcal{N}=8$ Supergravity contains a graviton $h_{\mu\nu}$ and its 8 gravitinos ψ^i_μ interacting with matter composed of 28 vectors $A^{[ij]}_\mu$, 56 spinors $\chi^{[ijk]}$, and 70 scalars $C^{[ijkl]}$, labelled with SO(8) indices, $i,j,k,l=1,2,\ldots,8$. The much larger Cremmer-Julia $E_{7(7)}$ symmetry acts on the scalars and the field strengths, and we begin with the manifestly SO(8) symmetric order-κ^2 $\mathcal{N}=8$ Supergravity Lagrangian [12] with those fields only. The scalar part is given by

$$\mathcal{L}_S = -\,\frac{1}{48}\left\{\partial_\mu C^{ijkl}\,\partial^\mu\overline{C}^{ijkl}+\frac{\kappa^2}{2}\,C^{ijkl}\,\overline{C}^{klmn}\,\partial_\mu C^{mnpq}\,\partial^\mu\overline{C}^{pqij}+\mathcal{O}(\kappa^3)\right\}, \tag{2.1}$$

where the scalar fields satisfy

$$C^{\,ijkl}=\frac{1}{4!}\,\epsilon^{ijklmnpq}\,\overline{C}^{\,mnpq}\ . \tag{2.2}$$

The Lagrangian with the field strengths is given by

$$\mathcal{L}_V\ =-\frac{1}{8}\,\mathcal{F}^{ij}_{\mu\nu}\,\mathcal{G}^{\mu\nu\,ij}+\ c.c.\ . \tag{2.3}$$

written in terms of the self-dual complex field strengths

$$\mathcal{F}^{\mu\nu\,ij}=\frac{1}{2}F^{\mu\nu\,ij}+\frac{i}{2}\widetilde{F}^{\mu\nu\,ij}\,, \tag{2.4}$$

and

$$\mathcal{G}^{\mu\nu\,ij}\ =\ \mathcal{F}^{\mu\nu\,ij}+\kappa\,\overline{C}^{ijkl}\mathcal{F}^{\mu\nu\,kl}+\frac{\kappa^2}{2}\overline{C}^{ijkl}\overline{C}^{klmn}\,\mathcal{F}^{\mu\nu\,mn}\,+\,\mathcal{O}(\kappa^3)\,, \tag{2.5}$$

is linear in the field strengths.

2.1 SU(8) and $E_{7(7)}$ dualities

The electro-magnetic duality transformations exchange equations of motion

$$\partial_\mu \left(\mathcal{G}^{\mu\nu\, ij} + \overline{\mathcal{G}}^{\mu\nu\, ij}\right) = 0\,, \tag{2.6}$$

for Bianchi identities

$$\partial_\mu \left(\mathcal{F}^{\mu\nu\, ij} - \overline{\mathcal{F}}^{\mu\nu\, ij}\right) = 0\ . \tag{2.7}$$

These equations are manifestly SO(8) covariant. We can elevate this symmetry to SU(8) [13] on the complex field strengths by demanding

$$\delta\,\left(\mathcal{G}^{\mu\nu\, ij} + \mathcal{F}^{\mu\nu\, ij}\right) = \left(R^{ik} + iS^{ik}\right)\left(\mathcal{G}^{\mu\nu\, kj} + \mathcal{F}^{\mu\nu\, kj}\right) - (i\ \leftrightarrow\ j)\,, \tag{2.8}$$

transforming as **28**, while the other combinations $(\mathcal{G}^{\mu\nu\, ij} - \mathcal{F}^{\mu\nu\, ij})$ transform as the complex conjugate $\overline{\mathbf{28}}$,

$$\delta\,\left(\mathcal{G}^{\mu\nu\, ij} - \mathcal{F}^{\mu\nu\, ij}\right) = \left(R^{ik} - iS^{ik}\right)\left(\mathcal{G}^{\mu\nu\, kj} - \mathcal{F}^{\mu\nu\, kj}\right) - (i\ \leftrightarrow\ j)\ . \tag{2.9}$$

where R^{ij} are the 28 *real* antisymmetric rotation tensors which generate SO(8), and S^{ij} are 35 *real* symmetric traceless matrices in the coset SU(8)/ SO(8). The SU(8)/ SO(8) coset transformations δ' on the complex field strengths

$$\delta'\,\mathcal{F}^{\mu\nu\, ij} = iS^{ik}\mathcal{G}^{\mu\nu\, kj} - (i\ \leftrightarrow\ j)\,, \qquad \delta'\,\mathcal{G}^{\mu\nu\, ij} = iS^{ik}\mathcal{F}^{\mu\nu\, kj} - (i\ \leftrightarrow\ j)\,, \tag{2.10}$$

are the duality transformations which map the equations of motion into the Bianchi identities and *vice-versa*

$$\delta'\left\{\partial_\mu(\mathcal{G}^{\mu\nu\, ij} + \overline{\mathcal{G}}^{\mu\nu\, ij})\right\} = i\,S^{ik}\,\partial_\mu(\mathcal{F}^{\mu\nu\, kj} - \overline{\mathcal{F}}^{\mu\nu\, kj})\ -\ (i \leftrightarrow j)\ . \tag{2.11}$$

The SU(8)/ SO(8) transformations are only symmetries of the equations of motion and the Bianchi identities, but not of the Lagrangian.

Consistency of the coset variation of this expression with the two variations of (2.10) requires that the scalar fields transform linearly under the full SU(8), that is

$$\delta'\,\overline{C}^{ijkl} = -\,i\,S^{im}\,\overline{C}^{mjkl}\ -\ (\,i \leftrightarrow j\,) - (\,i \leftrightarrow k\,) - (\,i \leftrightarrow l\,)\,, \tag{2.12}$$

i.e. as a **70**. This is an exact equation with no order κ corrections. It follows that the scalar Lagrangian (2.1) is SU(8) invariant. On the other hand, the complex field strengths have more complicated non-linear coset transformation

$$\begin{aligned}\delta'\,\mathcal{F}^{\mu\nu\, ij} = i\,S^{im}&\left(\mathcal{F}^{\mu\nu\, mj} + \kappa\overline{C}^{mjkl}\mathcal{F}^{\mu\nu\, kl}\ +\ \frac{\kappa^2}{2}\overline{C}^{mjkl}\overline{C}^{klpq}\,\mathcal{F}^{\mu\nu\, pq} + \mathcal{O}(\kappa^3)\right)\\ &-\ (i \leftrightarrow j)\ .\end{aligned} \tag{2.13}$$

The terms on the right-hand-side transform differently order by order in κ: $\mathcal{F}^{\mu\nu\, mj} \sim$ **28**, while $\overline{C}^{mjkl}\mathcal{F}^{\mu\nu\, kl} \sim \overline{\mathbf{28}}$, and the order κ^2 term has even more complicated coset transformations. Yet, one can check that the commutator of two such variations closes on SO(8) transformation, as required. The extension to SU(8) duality on the field strengths is meaningful only in the interacting case when $\kappa \neq 0$, since $\mathcal{G}^{\mu\nu\, ij} - \mathcal{F}^{\mu\nu\, ij} = \mathcal{O}(\kappa)$.

Cremmer and Julia extended the duality symmetries to the non-compact $E_{7(7)}$ [1]. Assemble the complex field strengths in one column vector with 56 complex entries [14]

$$Z^{\mu\nu} = \begin{pmatrix} \mathcal{G}^{\mu\nu\, ij} + \mathcal{F}^{\mu\nu\, ij} \\ \mathcal{G}^{\mu\nu\, ij} - \mathcal{F}^{\mu\nu\, ij} \end{pmatrix} \equiv \begin{pmatrix} X^{\mu\nu\, ab} \\ Y^{\mu\nu}{}_{ab} \end{pmatrix}, \tag{2.14}$$

where a, b are SU(8) indices, with upper(lower) antisymmetric indices for $\mathbf{28}(\overline{\mathbf{28}})$. Its two components

$$X^{\mu\nu\, ab} = 2\mathcal{F}^{\mu\nu\, ij} + \kappa \overline{C}^{ijkl} \mathcal{F}^{\mu\nu\, kl} + \frac{\kappa^2}{2} \overline{C}^{ijkl} \overline{C}^{klmn} \mathcal{F}^{\mu\nu\, mn} + \mathcal{O}(\kappa^3)\,, \tag{2.15}$$

$$Y^{\mu\nu}{}_{ab} = \kappa \overline{C}^{ijkl} \mathcal{F}^{\mu\nu\, kl} + \frac{\kappa^2}{2} \overline{C}^{ijkl} \overline{C}^{klmn} \mathcal{F}^{\mu\nu\, mn} + \mathcal{O}(\kappa^3)\,, \tag{2.16}$$

are *not independent*, but related by

$$Y^{\mu\nu}{}_{ab} - \frac{\kappa}{2} \overline{C}_{abcd} X^{\mu\nu\, cd} + \mathcal{O}(\kappa^2) = 0\,. \tag{2.17}$$

The equations of motion (2.6) and Bianchi identities (2.7) can be written in terms of $Z^{\mu\nu}$

$$\partial_\mu \left(Z^{\mu\nu} + \widetilde{Z}^{\mu\nu} \right) = 0\,, \tag{2.18}$$

where

$$\widetilde{Z}^{\mu\nu} \equiv \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \overline{Z}^{\mu\nu} = \begin{pmatrix} \overline{\mathcal{G}}^{\mu\nu\, ij} - \overline{\mathcal{F}}^{\mu\nu\, ij} \\ \overline{\mathcal{G}}^{\mu\nu\, ij} + \overline{\mathcal{F}}^{\mu\nu\, ij} \end{pmatrix} = \begin{pmatrix} \overline{Y}^{\mu\nu\, ab} \\ \overline{X}^{\mu\nu}{}_{ab} \end{pmatrix}.$$

The upper component of (2.18) is the sum of the equations of motion and the Bianchi identities, and the lower component the difference. It follows that the duality transformations are those which act the same way on both $Z^{\mu\nu}$ and $\widetilde{Z}^{\mu\nu}$. Explicitly, under the coset transformation denoted by $\boldsymbol{\delta}$

$$\boldsymbol{\delta}\, X^{\mu\nu\, ab} = \Xi^{abcd}\, Y^{\mu\nu}{}_{cd}\,, \tag{2.19}$$

$$\boldsymbol{\delta}\, Y^{\mu\nu}{}_{ab} = \overline{\Xi}_{abcd}\, X^{\mu\nu\, cd}\,, \tag{2.20}$$

transform $\mathbf{28}$ into $\overline{\mathbf{28}}$ and vice versa. It can be checked that such transformations with real Ξ^{abcd} leave both equations of motion and Bianchi identities invariant, while those with pure imaginary Ξ^{abcd} are duality transformations which interchange the two. The transformations must respect the constraint (2.17) between the upper and lower components of $Z^{\mu\nu}$

$$\boldsymbol{\delta} Y^{\mu\nu}{}_{ab} = \frac{\kappa}{2} \boldsymbol{\delta} \left(\overline{C}_{abcd} X^{\mu\nu\, ab} \right) + \mathcal{O}(\kappa^2)\,,$$

that is

$$\overline{\Xi}_{abcd} X^{\mu\nu\, cd} = \frac{\kappa}{2}\, \boldsymbol{\delta} \overline{C}_{abcd} X^{\mu\nu\, cd} + \frac{\kappa}{2} \overline{C}_{abef} \Xi^{efmn} (\frac{\kappa}{2} \overline{C}_{mncd} X^{\mu\nu\, cd}) + \mathcal{O}(\kappa^2)\,.$$

It follows that the scalars must transform non-linearly as

$$\boldsymbol{\delta}\overline{C}_{abcd} = \frac{2}{\kappa}\Xi_{abcd} - \frac{\kappa}{2}\overline{C}_{ef[ab}\,\overline{C}_{cd]mn}\Xi^{efmn} + \mathcal{O}(\kappa^3)\,, \tag{2.21}$$

where the indices inside the square brackets are antisymmetrized. Since the scalars satisfy the self duality condition (2.2), so must Ξ^{abcd}

$$\Xi^{abcd} = \frac{1}{4!}\,\epsilon^{abcdefgh}\,\overline{\Xi}_{efgh}\,, \tag{2.22}$$

which restricts Ξ^{abcd} to 70 real parameters. It also means that the extra term in (2.21) is self-dual. Repeated use of (2.20) yields the commutator

$$[\,\boldsymbol{\delta}_1,\,\boldsymbol{\delta}_2]\,X^{\mu\nu\,ab} = \left(\Xi^{abef}_{(2)}\,\overline{\Xi}_{(1)efcd} - \Xi^{abef}_{(1)}\,\overline{\Xi}_{(2)\,efcd}\right)X^{\mu\nu\,cd}\,.$$

We can show [11] that the duality requirement (2.22) on the parameters of this commutator yields exactly the 63 parameters of SU(8), resulting in a 133-parameter group, the non-compact $E_{7(7)}$ since the $E_{7(7)}/\,\mathrm{SU}(8)$ transformations are not unitary. The $E_{7(7)}/\,\mathrm{SU}(8)$ transformations of the complex field strengths follow

$$\begin{aligned}\boldsymbol{\delta}\,\mathcal{F}^{\mu\nu\,ij} &= -\,\overline{\Xi}^{ijkl}\mathcal{F}^{\mu\nu\,kl} + \frac{\kappa}{2}\left(\Xi^{ijkl} - \overline{\Xi}^{ijkl}\right)\overline{C}^{klmn}\mathcal{F}^{\mu\nu\,mn}\\ &\quad + \frac{\kappa^2}{4}\left(\Xi^{ijkl} - \overline{\Xi}^{ijkl}\right)\overline{C}^{klmn}\overline{C}^{mnpq}\mathcal{F}^{\mu\nu\,pq} + \mathcal{O}(\kappa^3)\,.\end{aligned} \tag{2.23}$$

As we mentioned before, this equation is meaningful only when $\kappa \neq 0$. While the scalar part of the Lagrangian $\mathcal{L}_S$ is $E_{7(7)}$-invariant, the vector Lagrangian $\mathcal{L}_V$ is not. Invariance is attained only after invoking the equations of motion.

3. $E_{7(7)}$ invariance on the light-cone

The Abelian field strengths are written in terms of the potentials A^{ij}_μ through

$$F^{\mu\nu\,ij} = \partial^\mu A^{\nu\,ij} - \partial^\nu A^{\mu\,ij}\,.$$

In the LC_2 formalism we choose the gauge conditions

$$A^{+\,ij} = \frac{1}{\sqrt{2}}\left(A^0 + A^3\right)^{ij} = 0\,, \tag{3.1}$$

and invert the equations of motion to express $A^{-\,ij}$ in terms of the remaining variables in the theory, the physical transverse components of the *complex* vector potentials

$$\bar{A}^{ij} = \frac{1}{\sqrt{2}}(A^1\,+\,i\,A^2)^{ij}\,;\qquad A^{ij} = \frac{1}{\sqrt{2}}(A^1\,-\,i\,A^2)^{ij}\,.$$

A lengthy but straightforward computation yields

$$\begin{aligned}
A^{-\,ij} &\equiv \frac{1}{\sqrt{2}}\left(A^0 - A^3\right)^{ij} \\
&= \frac{\partial}{\partial^+}\,A^{ij} + \frac{\overline{\partial}}{\partial^+}\,\bar{A}^{ij} - \kappa\frac{1}{\partial^+}\left(\overline{C}^{ijkl}\partial\,A^{kl}\right) - \kappa\frac{1}{\partial^+}\left(C^{ijkl}\overline{\partial}\,\bar{A}^{kl}\right) \\
&\quad + \kappa\frac{\partial}{\partial^{+2}}\,(\,\overline{C}^{ijkl}\partial^+\,A^{kl}\,) + \kappa\frac{\overline{\partial}}{\partial^{+2}}(\,C^{ijkl}\partial^+\,\bar{A}^{kl}\,) \\
&\quad + \frac{\kappa^2}{2}\,\frac{1}{\partial^+}\,\Big[\,C^{ijkl}\overline{C}^{klmn}\partial A^{mn} + \overline{C}^{ijkl}C^{klmn}\bar{\partial}\bar{A}^{mn} \\
&\quad -(C^{ijkl}+\overline{C}^{ijkl})\frac{\partial}{\partial^+}(\overline{C}^{klmn}\partial^+A^{mn}) - (C^{ijkl}+\overline{C}^{ijkl})\frac{\bar{\partial}}{\partial^+}(C^{klmn}\partial^+\bar{A}^{mn}) \\
&\quad + \frac{\partial}{\partial^+}\,(\overline{C}^{ijkl}\overline{C}^{klmn}\partial^+A^{mn}\,) + \frac{\bar{\partial}}{\partial^+}(\,C^{ijkl}C^{klmn}\partial^+\bar{A}^{mn})\Big] + \mathcal{O}(\kappa^3)\,,
\end{aligned} \tag{3.2}$$

where

$$\bar{\partial} = \frac{1}{\sqrt{2}}\,(\,\partial_1\,-\,i\,\partial_2\,)\,,\qquad \partial = \frac{1}{\sqrt{2}}\,(\,\partial_1\,+\,i\,\partial_2)\,,\qquad \partial^+ = \frac{1}{\sqrt{2}}\,(\,-\partial_0+\partial_3\,)$$

(The occurrence of the non-local operator $\frac{1}{\partial^+}$ is abundant in the LC_2 formalism. It is a harmless non-locality along the light-cone which is well understood).

This enables us to find the LC_2 complex field strengths $\mathcal{F}^{+-ij}$

$$\begin{aligned}
\mathcal{F}^{+-ij} &= \frac{1}{2}\left(\partial^+A^{-\,ij} + \partial A^{ij} - \bar{\partial}\bar{A}^{ij}\right) \\
&= \partial A^{ij}\; - \frac{\kappa}{2}\overline{C}^{ijkl}\partial A^{kl} - \frac{\kappa}{2}C^{ijkl}\bar{\partial}\bar{A}^{kl} + \frac{\kappa}{2}\frac{\partial}{\partial^+}(\overline{C}^{ijkl}\partial^+A^{kl}) + \frac{\kappa}{2}\frac{\bar{\partial}}{\partial^+}(C^{ijkl}\partial^+\bar{A}^{kl}) \\
&\quad + \frac{\kappa^2}{4}\Big[C^{ijkl}\overline{C}^{klmn}\partial A^{mn} + \overline{C}^{ijkl}C^{klmn}\bar{\partial}\bar{A}^{mn} \\
&\qquad -(C^{ijkl}+\overline{C}^{ijkl})\left(\frac{\partial}{\partial^+}(\overline{C}^{klmn}\partial^+A^{mn}) + \frac{\bar{\partial}}{\partial^+}(C^{klmn}\partial^+\bar{A}^{mn})\right) \\
&\qquad + \frac{\partial}{\partial^+}\left(\overline{C}^{ijkl}\overline{C}^{klmn}\partial^+A^{mn}\right) + \frac{\bar{\partial}}{\partial^+}\left(C^{ijkl}C^{klmn}\partial^+\bar{A}^{mn}\right)\Big] + \cdots\,.
\end{aligned} \tag{3.3}$$

By varying this expression and using (2.13), we arrive at the non-linear transformation of the physical vector potentials under SU(8)/ SO(8)

$$\delta'\,A^{ij} = iS^{im}\left(A^{mj} + \kappa\frac{1}{\partial^+}\left(\overline{C}^{mjkl}\partial^+A^{kl}\right) + \mathcal{O}(\kappa^3)\right) - \;(i\leftrightarrow j)\;. \tag{3.4}$$

As in the covariant case, the terms on the right-hand-side do not share the same coset transformations.

Similarly, the coset $E_{7(7)}/\,\mathrm{SU}(8)$ transformations of the vector potentials are obtained by substituting $\mathcal{F}^{+-ij}$ in (2.23) with (3.3). Remembering that the scalars transform non-linearly under $E_{7(7)}/\,\mathrm{SU}(8)$ (2.21) , we find for the vector potentials

$$\boldsymbol{\delta} A^{ij} \;=\; -\,\overline{\Xi}^{ijkl}A^{kl} + \frac{\kappa}{2}(\Xi^{ijkl} - \overline{\Xi}^{ijkl})\frac{1}{\partial^+}\left(\overline{C}^{klmn}\partial^+A^{mn}\right) + \mathcal{O}(\kappa^3)\,, \tag{3.5}$$

which preserve helicity, and exist as long as $\kappa \neq 0$.

3.1 The vector and scalar LC_2 hamiltonians

The vector Lagrangian (2.3) in the LC_2 gauge, obtained by setting $A^{+\,ij} = 0$ and replacing $A^{-\,ij}$ using the equations of motion, is given by

$$
\begin{aligned}
\mathcal{L}_V \,=\, & \bar{A}^{ij}\,(\,-\partial^+\,\partial^- \,+\, \partial\,\bar{\partial}\,)\,A^{ij} \,+\, \frac{\kappa}{2}\Big[\partial^+ A^{ij}\,\overline{C}^{ijmn}\,\partial^-\bar{A}^{mn} \\
& + \partial A^{ij}\left(\overline{C}^{ijkl}\partial A^{kl} - \frac{\partial}{\partial^+}(\overline{C}^{ijkl}\partial^+ A^{kl}) - \frac{\bar{\partial}}{\partial^+}(C^{ijkl}\partial^+\bar{A}^{kl})\right) \,+\, c.c.\Big] \\
& - \frac{\kappa^2}{2}\frac{\bar{\partial}}{\partial^+}(\partial^+\bar{A}^{ij}\,C^{ijkl})\,\frac{\partial}{\partial^+}(\overline{C}^{klmn}\partial^+ A^{mn}) \\
& + \frac{\kappa^2}{2}\,\Big[-\frac{1}{2}\partial A^{ij}\overline{C}^{ijkl}C^{klmn}\bar{\partial}\bar{A}^{mn} \,+\, \partial A^{ij}\overline{C}^{ijkl}\,\frac{\bar{\partial}}{\partial^+}(C^{klmn}\partial^+\bar{A}^{mn}) \,+\, c.c.\Big] \\
& + \frac{\kappa^2}{2}\,\Big[\partial A^{ij}\overline{C}^{ijkl}\frac{\partial}{\partial^+}(\overline{C}^{klmn}\partial^+ A^{mn}) - \frac{1}{2}\frac{\partial}{\partial^+}(\partial^+ A^{ij}\overline{C}^{ijkl})\frac{\partial}{\partial^+}(\overline{C}^{klmn}\partial^+ A^{mn}) + c.c.\Big] \\
& + \frac{\kappa^2}{4}\,\Big[\partial^-\bar{A}^{ij}\overline{C}^{ijkl}\overline{C}^{klmn}\partial^+ A^{mn} \,-\, \frac{\partial}{\partial^+}(\partial^+ A^{ij}\overline{C}^{ijkl}\overline{C}^{klmn})\,\big(\partial A^{mn} + \bar{\partial}\bar{A}^{mn}\big) + c.c.\Big] \\
& + \mathcal{O}(\kappa^3)\,,
\end{aligned}
\tag{3.6}
$$

while the scalar Supergravity Lagrangian (2.1) becomes

$$
\begin{aligned}
\mathcal{L}_S \,=\, & -\frac{1}{24}C^{ijkl}(\partial^+\partial^- - \partial\bar{\partial})\overline{C}^{ijkl} \\
& + \frac{\kappa^2}{96}\,C^{ijkl}\,\overline{C}^{klmn}\,(\partial^+ C^{mnpq}\,\partial^-\overline{C}^{pqij} + \partial^- C^{mnpq}\,\partial^+\overline{C}^{pqij} \\
& \qquad - \partial C^{mnpq}\,\bar{\partial}\overline{C}^{pqij} - \bar{\partial}C^{mnpq}\,\partial\overline{C}^{pqij}) + \mathcal{O}(\kappa^3)\;.
\end{aligned}
\tag{3.7}
$$

Both contain the light-cone time derivative ∂^- in their interactions. In order to have a Hamiltonian without this derivative we eliminate it by the field redefinitions

$$
\begin{aligned}
C^{\,ijkl} \,=\, & D^{\,ijkl} \,-\, \frac{\kappa^2}{4}\,\frac{1}{\partial^+}\left(D^{pq[ij}\partial^+ D^{kl]mn}\overline{D}_{pqmn}\right) + \\
& + \frac{3\,\kappa^2}{2\,\partial^+}\left(\partial^+ B^{[ij}\frac{1}{\partial^+}(D^{kl]mn}\partial^+\overline{B}_{mn})\right) + \\
& + \frac{3\,\kappa^2}{2\cdot 4!\,\partial^+}\epsilon^{ijklrstu}\left(\partial^+\overline{B}_{rs}\frac{1}{\partial^+}(\overline{D}_{tumn}\partial^+ B^{mn})\right) \,+\, \mathcal{O}(\kappa^3)\,,
\end{aligned}
\tag{3.8}
$$

$$
A^{ij} \,=\, B^{ij} - \frac{\kappa}{2}\frac{1}{\partial^+}\left(\overline{D}_{ijkl}\partial^+ B^{kl}\right) + \frac{\kappa^2}{8}D^{ijkl}\frac{1}{\partial^+}\left(\partial^+ B^{mn}\overline{D}_{mnkl}\right) + \mathcal{O}(\kappa^3)\;. \tag{3.9}
$$

This procedure leads to the unique Hamiltonian of the theory in component form.

The new vector potentials, B^{ij} now transform *linearly* under SU(8),

$$
\delta'\,B^{ij} = iS^{ik}\,B^{kj} - (i \leftrightarrow j)\,,
$$

so that $i, j, \ldots$ are now true SU(8) indices; in particular, their lowering produces the "barred" representation. The $E_{7(7)}/\,\mathrm{SU}(8)$ variations of the redefinitions yield the transformation properties of the new fields

$$\begin{aligned}
\boldsymbol{\delta}\, B^{ij} &= -\frac{\kappa}{4}\,\overline{\Xi}_{mnkl} D^{ijkl} B^{mn} + \frac{\kappa}{4}\,\Xi^{ijkl}\frac{1}{\partial^+}\left(\overline{D}_{mnkl}\partial^+ B^{mn}\right) + \mathcal{O}(\kappa^3)\,, \qquad (3.10)\\
\boldsymbol{\delta} D^{ijkl} &= \frac{2}{\kappa}\,\Xi^{ijkl} - \frac{\kappa}{2}\,\overline{\Xi}_{mnpq}\frac{1}{\partial^+}\left(D^{mn[kl}\partial^+ D^{ij]pq}\right)\\
&\quad + \frac{\kappa}{2}\,\Xi^{pq[ij}\frac{1}{\partial^+}\left(\partial^+ D^{kl]mn}\overline{D}_{pqmn}\right)\\
&\quad - 3\,\kappa\left(\frac{\Xi^{mn[kl}}{\partial^+}\left(\partial^+ B^{ij]}\overline{B}_{mn}\right) + \epsilon^{ijklrstu}\frac{\overline{\Xi}_{tumn}}{4!\partial^+}\left(B^{mn}\partial^+\overline{B}_{rs}\right)\right)\\
&\quad + \mathcal{O}(\kappa^3)\,. \qquad (3.11)
\end{aligned}$$

We note that the $E_{7(7)}/\,\mathrm{SU}(8)$ variation of the scalars contains terms quadratic in the gauge fields. This mixing does not occur in the covariant formalism. Complicated as they may seem, these variations are still incomplete since they do not include the other fields of the theory. We will use the Supersymmetry of $\mathcal{N} = 8$ Supergravity to generalize the transformations (3.10) and (3.11) to include them, and defer to a later publication the construction of the vector and scalar Hamiltonians in component form, as well as the proof of their $E_{7(7)}$ invariance [11].

4. $\mathcal{N} = 8$ supergravity in light-cone superspace

The 256 physical degrees of freedom of $\mathcal{N} = 8$ Supergravity form *one* constrained chiral superfield in the superspace spanned by eight Grassmann variables, θ^m and their complex conjugates $\bar\theta_m$ ($m = 1, \ldots, 8$), on which SU(8) acts linearly. We introduce the chiral derivatives

$$d^m \equiv -\frac{\partial}{\partial\bar\theta_m} - \frac{i}{\sqrt{2}}\theta^m\partial^+\,, \quad \bar d_m \equiv \frac{\partial}{\partial\theta^m} + \frac{i}{\sqrt{2}}\bar\theta_m\partial^+\,, \qquad (4.1)$$

which satisfy canonical anticommutation relations

$$\left\{d^m\,,\,\bar d_n\right\} = -i\sqrt{2}\delta^m{}_n\partial^+\ . \qquad (4.2)$$

The physical degrees of freedom of $\mathcal{N} = 8$ Supergravity, the spin-2 graviton h and $\overline{h}$; eight spin-$\frac{3}{2}$ gravitinos, ψ^m and $\overline{\psi}_m$, twenty eight vector fields

$$\overline{B}_{mn} \equiv \frac{1}{\sqrt{2}}\left(B^1_{mn} + i\,B^2_{mn}\right)\,,$$

and their conjugates, fifty six gauginos $\overline{\chi}_{mnp}$ and χ^{mnp}, and finally seventy real scalars $\overline{D}_{mnpq}$ appear in one superfield

$$\begin{aligned}
\varphi\,(\,y\,) &= \frac{1}{\partial^{+2}}\,h\,(y) + i\,\theta^m\,\frac{1}{\partial^{+2}}\,\overline{\psi}_m\,(y) + i\,\theta^{mn}\,\frac{1}{\partial^+}\,\overline{B}_{mn}\,(y)\\
&\quad - \theta^{mnp}\,\frac{1}{\partial^+}\,\overline{\chi}_{mnp}\,(y) - \theta^{mnpq}\,\overline{D}_{mnpq}\,(y) + i\widetilde{\theta}_{\ mnp}\,\chi^{mnp}\,(y)\\
&\quad + i\widetilde{\theta}_{\ mn}\,\partial^+\,B^{mn}\,(y) + \widetilde{\theta}_{\ m}\,\partial^+\,\psi^m\,(y) + 4\,\widetilde{\theta}\,\partial^{+2}\,\bar h\,(y)\,, \qquad (4.3)
\end{aligned}$$

where the bar denotes complex conjugation, and

$$\theta^{a_1a_2\ldots a_n} = \frac{1}{n!}\,\theta^{a_1}\theta^{a_2}\cdots\theta^{a_n}\,, \quad \widetilde{\theta}_{\;a_1a_2\ldots a_n} = \epsilon_{a_1a_2\ldots a_nb_1b_2\ldots b_{(8-n)}}\,\theta^{b_1b_2\cdots b_{(8-n)}}\;.$$

The arguments of the fields are the chiral coordinates

$$y = (x,\,\bar{x},\,x^+,\,y^-\equiv x^- - \frac{i}{\sqrt{2}}\theta^m\bar{\theta}_m\,)\,, \qquad x = \frac{1}{\sqrt{2}}(x_1+ix_2)\,,$$

so that φ and its complex conjugate $\overline{\varphi}$ satisfy the chiral constraints

$$d^m\,\varphi = 0, \qquad \overline{d}_m\,\overline{\varphi} = 0\,, \tag{4.4}$$

The complex chiral superfield is related to its complex conjugate by the *inside-out constraint*

$$\varphi = \frac{1}{4\,\partial^{+4}}\,d^1d^2\cdots d^8\,\overline{\varphi}\,, \tag{4.5}$$

in accordance with the duality condition of D^{mnpq}.

On the light-cone, the eight kinematical supersymmetries (the spectrum-generating part of the symmetry) are linearly represented by the operators q^m and $\bar{q}_m$

$$q^m = -\frac{\partial}{\partial\bar{\theta}_m} + \frac{i}{\sqrt{2}}\theta^m\partial^+\,, \quad \bar{q}_m = \frac{\partial}{\partial\theta^m} - \frac{i}{\sqrt{2}}\bar{\theta}_m\partial^+\,, \tag{4.6}$$

which also satisfy anticommutation relation

$$\{\,q^m\,,\,\bar{q}_n\,\} = i\sqrt{2}\,\delta^m{}_n\,\partial^+\,, \tag{4.7}$$

and anticommute with the chiral derivatives. Hence, their linear action on the chiral superfield

$$\delta_s\,\varphi(y) = \overline{\epsilon}_m\,q^m\,\varphi(y)\,, \tag{4.8}$$

where $\overline{\epsilon}_m$ is the parameter of the supersymmetry transformation, preserves chirality. The kinematical supersymmetry transformations of the physical fields are then

$$\begin{aligned}
\delta_s\,h &= 0\,, & \delta_s\,\overline{h} &= -i\frac{\sqrt{2}}{4}\,\overline{\epsilon}_m\,\psi^m\,,\\
\delta_s\,\psi^m &= 2\sqrt{2}\,\overline{\epsilon}_n\partial^+\,B^{mn}\,, & \delta_s\,\overline{\psi}_m &= -\sqrt{2}\,\overline{\epsilon}_m\,\partial^+\,h\,,\\
\delta_s\,B^{mn} &= -3i\sqrt{2}\,\overline{\epsilon}_p\,\chi^{mnp}\,, & \delta_s\,\overline{B}_{mn} &= -2i\sqrt{2}\,\overline{\epsilon}_{[m}\overline{\psi}_{n]}\,,\\
\delta_s\,\chi^{lmn} &= -\frac{\sqrt{2}}{3!}\,\overline{\epsilon}_k\,\partial^+D^{klmn}\,, & \delta_s\,\overline{\chi}_{mnp} &= -3\sqrt{2}\,\epsilon_{[p}\,\partial^+\,\overline{B}_{mn]}\,,
\end{aligned}$$

and finally

$$\delta_s\,\overline{D}_{klmn} = -4i\sqrt{2}\,\overline{\epsilon}_{[n}\,\overline{\chi}_{klm]}\;.$$

The quadratic operators

$$T^i{}_j = -\frac{i}{\sqrt{2}\,\partial^+}\left(q^i\bar{q}_j - \frac{1}{8}\delta^i{}_jq^k\bar{q}_k\right)\,, \tag{4.9}$$

which satisfy the SU(8) algebra

$$[\,T^i{}_j\,,\,T^k{}_l\,] = \delta^k{}_j\,T^i{}_l - \delta^i{}_l\,T^k{}_j\,,$$

also act linearly on the chiral superfield

$$\delta_{SU_8}\,\varphi(y) = \omega^j{}_i\,T^i{}_j\,\varphi(y)\ .$$

We can now include the other fields of the theory by demanding that the $E_{7(7)}/\,\mathrm{SU}(8)$ transformations commute with the kinematical supersymmetries, that is

$$[\,\delta_s\,,\,\boldsymbol{\delta}\,]\,\varphi(y) = 0\ . \tag{4.10}$$

We begin by applying this equation to the vector potential. By carefully choosing the parameters of both supersymmetry and of $E_{7(7)}/\,\mathrm{SU}(8)$, we arrive at the generalization of (3.10) to order κ

$$\begin{aligned}
\boldsymbol{\delta}\,\overline{B}_{ij} \;=\; & -\kappa\,\Xi^{klmn}\left(\frac{1}{4}\overline{D}_{ijkl}\,\overline{B}_{mn} + \frac{1}{4!}\frac{1}{\partial^+}\overline{D}_{klmn}\partial^+\overline{B}_{ij} - \frac{1}{4!}\epsilon_{ijklmnrs}\frac{1}{\partial^+}B^{rs}\partial^+h \right.\\
& \left. + \frac{i}{3!}\frac{1}{\partial^+}\overline{\chi}_{klm}\,\overline{\chi}_{ijn} - \frac{i}{3!}\,\epsilon_{ijklmrst}\frac{1}{\partial^+}\chi^{rst}\overline{\psi}_n\right)\\
& + \kappa\,\overline{\Xi}_{ijkl}\,\frac{1}{\partial^+}\left(\frac{1}{4}\,D^{klmn}\,\partial^+\overline{B}_{mn} - \frac{1}{\partial^+}\,B^{kl}\,\partial^{+2}\,h \right.\\
& \left. + \frac{i}{4(3!)^2}\overline{\chi}_{mnp}\overline{\chi}_{rst}\epsilon^{klmnprst} - 3\,i\,\frac{1}{\partial^+}\chi^{kln}\partial^+\overline{\psi}_n\right)\\
& + \mathcal{O}(\kappa^3)\,,
\end{aligned} \tag{4.11}$$

as well as to the $E_{7(7)}/\,\mathrm{SU}(8)$ transformations of the gravitinos since commutativity implies

$$\delta_s\,\boldsymbol{\delta}\,\overline{B}_{ij} = -2i\sqrt{2}\overline{\epsilon}_{[i}\,\boldsymbol{\delta}\,\overline{\psi}_{j]}\ .$$

The result is

$$\begin{aligned}
\boldsymbol{\delta}\,\overline{\psi}_i \;=\; & -\kappa\,\Xi^{mnpq}\left(\frac{1}{4!}\frac{1}{\partial^+}\overline{D}_{mnpq}\partial^+\overline{\psi}_i + \frac{1}{3!}\overline{D}_{mnpi}\overline{\psi}_q \right.\\
& \left. - \frac{1}{4!}\epsilon_{mnpqirst}\frac{1}{\partial^+}\chi^{rst}\partial^+h + \frac{1}{4}\overline{\chi}_{imn}\overline{B}_{pq} + \frac{1}{3!}\frac{1}{\partial^+}\overline{\chi}_{mnp}\partial^+\overline{B}_{iq}\right)\\
& + \mathcal{O}(\kappa^3)\ .
\end{aligned} \tag{4.12}$$

Applying commutativity on the gravitinos yields the $E_{7(7)}/\,\mathrm{SU}(8)$ transformation of the graviton

$$\delta_s\,\boldsymbol{\delta}\,\overline{\psi}_i = -\sqrt{2}\,\overline{\epsilon}_i\,\partial^+\,\boldsymbol{\delta}\,h\,,$$

with

$$\boldsymbol{\delta}\,h = -\kappa\,\Xi^{ijkl}\left(\frac{1}{8}\overline{B}_{ij}\overline{B}_{kl} + \frac{1}{4!}\frac{1}{\partial^+}\overline{D}_{ijkl}\partial^+h + \frac{i}{6}\frac{1}{\partial^+}\overline{\chi}_{ijk}\overline{\psi}_l\right) + \mathcal{O}(\kappa^3)\ . \tag{4.13}$$

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All these transformations are non-linear. Similar equations can be derived for the fifty six spinors and seventy scalars.

The inhomogeneous $E_{7(7)}/\,\mathrm{SU}(8)$ transformations of order κ^{-1} of the scalar fields can be expressed in superfield language, that is

$$\boldsymbol{\delta}^{(-1)}\,\varphi = -\frac{2}{\kappa}\,\theta^{ijkl}\,\overline{\Xi}_{ijkl}\,,$$

which is chiral since $E_{7(7)}$ is a global symmetry: $\partial^+\overline{\Xi}_{ijkl} = 0$. The order κ transformations of the superfield itself take a particularly simple form. We need only require that its variation be chiral, with the tensor structure

$$\kappa\,\Xi^{ijkl}(\cdots)_{ijkl}\ .$$

Assuming that the lower indices are carried by the antichiral derivatives $\overline{d}_n$ leads to the unique form of the transformation to first order in κ

$$\frac{\kappa}{4!}\,\Xi^{ijkl}\,\frac{1}{\partial^{+2}}\left(\overline{d}_{ijkl}\frac{1}{\partial^+}\varphi\,\partial^{+3}\varphi\,-\,4\,\overline{d}_{ijk}\varphi\,\overline{d}_l\partial^{+2}\varphi\,+\,3\,\overline{d}_{ij}\partial^+\varphi\,\overline{d}_{kl}\partial^+\varphi\right)\,,$$

where $\bar{d}_{k\ldots l}$ is a shorthand notation for $\bar{d}_k\,\ldots\,\bar{d}_l$. Including the inhomogeneous term, the $E_{7(7)}/\,\mathrm{SU}(8)$ transformation can be written in a more compact way by introducing a coherent state-like representation

$$\boldsymbol{\delta}\,\varphi = -\frac{2}{\kappa}\,\theta^{ijkl}\,\overline{\Xi}_{ijkl}\,+\,\frac{\kappa}{4!}\,\Xi^{ijkl}\left(\frac{\partial}{\partial\eta}\right)_{ijkl}\frac{1}{\partial^{+2}}\left(e^{\eta\hat{\bar{d}}}\partial^{+3}\varphi\,e^{-\eta\hat{\bar{d}}}\partial^{+3}\varphi\right)\bigg|_{\eta=0}\,+\,\mathcal{O}(\kappa^3)\,,\quad(4.14)$$

where

$$\eta\hat{\bar{d}} = \eta^m\frac{\bar{d}_m}{\partial^+},\quad\text{and}\quad\left(\frac{\partial}{\partial\eta}\right)_{ijkl}\;\equiv\;\frac{\partial}{\partial\eta^i}\frac{\partial}{\partial\eta^j}\frac{\partial}{\partial\eta^k}\frac{\partial}{\partial\eta^l}\ .$$

We note that these $E_{7(7)}/\,\mathrm{SU}(8)$ transformations do close properly to an SU(8) transformation on the superfield

$$[\,\boldsymbol{\delta}_1\,,\,\boldsymbol{\delta}_2\,]\,\varphi = \delta_{\mathrm{SU}(8)}\,\varphi\ .$$

It is chiral by construction $d^n\,\boldsymbol{\delta}\varphi = 0$, with the power of the first inverse derivative set by comparing with the graviton transformation. Hence, *all* physical fields, including the graviton transform under $E_{7(7)}$ and can be read off from this equation. It will be interesting to see what constraints this puts on the geometry.

We can now extend the method to the dynamical supersymmetries, and determine the form of the interactions implied by the $E_{7(7)}$ symmetry.

4.1 Superspace action

The $\mathcal{N} = 8$ Supergravity action in superspace was first obtained in [16] and its LC_2 form is derived in [17] to order κ, using algebraic consistency and simplified further in [18]. It is remarkably simple:

$$S = -\frac{1}{64}\int d^4x\int d^8\theta\,d^8\overline{\theta}\,\left\{-\,\overline{\varphi}\,\frac{\Box}{\partial^{+4}}\,\varphi\,-\,2\kappa\left(\frac{1}{\partial^{+2}}\overline{\varphi}\,\overline{\partial}\varphi\,\overline{\partial}\varphi\,+\,c.c.\right)\,+\,\mathcal{O}(\kappa^2)\right\}\,,\quad(4.15)$$

where $\Box \equiv 2\,(\,\partial\bar\partial\;-\;\partial^+\partial^-)$. The light-cone superfield Hamiltonian density is then written as

$$\mathcal{H} = 2\,\overline{\varphi}\,\frac{\partial\bar\partial}{\partial^{+4}}\,\varphi \,+\, 2\,\kappa\left(\frac{1}{\partial^{+2}}\overline{\varphi}\,\overline{\partial}\varphi\,\overline{\partial}\varphi \,+\, c.c.\right) \,+\, \mathcal{O}(\kappa^2)\;. \tag{4.16}$$

It can be derived from the action of the dynamical supersymmetries on the chiral superfield

$$\begin{aligned}\delta_s^{dyn}\,\varphi &= \delta_s^{dyn\,(0)}\,\varphi \,+\, \delta_s^{dyn\,(1)}\,\varphi \,+\, \delta_s^{dyn\,(2)}\,\varphi \,+\, \mathcal{O}(\kappa^3)\,, \qquad\qquad (4.17)\\ &= \epsilon^m\left\{\frac{\partial}{\partial^+}\,\bar q_m\,\varphi + \kappa\,\frac{1}{\partial^+}\left(\bar\partial\,\bar d_m\,\varphi\,\partial^{+2}\,\varphi \,-\, \partial^+\,\bar d_m\,\varphi\,\partial^+\,\bar\partial\,\varphi\right) + \mathcal{O}(\kappa^2)\right\}\;.\end{aligned}$$

We now require that the $E_{7(7)}/\,\mathrm{SU}(8)$ commutes with the dynamical supersymmetries

$$[\,\boldsymbol{\delta}\,,\,\delta_s^{dyn}\,]\,\varphi = 0\;. \tag{4.18}$$

This commutativity is valid only on the chiral superfield. For example, $[\,\boldsymbol{\delta}_1\,,\,\delta_s\,]\,\boldsymbol{\delta}_2\varphi \neq 0$, due to the non-linearity of the $E_{7(7)}$ transformation. This helps us understand how the Jacobi identity

$$(\,[\,\boldsymbol{\delta}_1\,,\,[\,\boldsymbol{\delta}_2\,,\,\delta_s\,]\,] \,+\, [\,\boldsymbol{\delta}_2\,,\,[\,\delta_s\,,\,\boldsymbol{\delta}_1\,]\,] \,+\, [\,\delta_s\,,\,[\,\boldsymbol{\delta}_1\,,\,\boldsymbol{\delta}_2\,]\,]\,)\,\varphi \;=\; 0\,,$$

is algebraically consistent. In the last term the commutator of the two $E_{7(7)}/\,\mathrm{SU}(8)$ transformations, $[\,\boldsymbol{\delta}_1\,,\,\boldsymbol{\delta}_2\,]$, yields an SU(8) under which the supersymmetry transforms. This is precisely compensated by contributions from the first two terms.

Although the dynamical supersymmetry to order κ is already known, we re-derive $\delta_s^{dyn\,(1)}\,\varphi$ from the commutativity between the dynamical supersymmetries and $E_{7(7)}/\,\mathrm{SU}(8)$ transformations.

The inhomogeneous $E_{7(7)}$ transformations link interaction terms with different order in κ. To zeroth order, one finds

$$[\,\boldsymbol{\delta}^{(-1)}\,,\,\delta_s^{dyn\,(1)}\,]\,\varphi = \boldsymbol{\delta}^{(-1)}\,\delta_s^{dyn\,(1)}\varphi = 0\,, \tag{4.19}$$

since $\delta_s^{dyn\,(1)}\,\boldsymbol{\delta}^{(-1)}\varphi = 0$. To find $\delta_s^{dyn\,(1)}\varphi$ that satisfies both the above equation and the SuperPoincaré algebra, one may start with a general form that satisfies all the commutation relations with the kinematical SuperPoincaré generators (the forms of the kinematical SuperPoincaré generators can be found in [19]),

$$\delta_s^{dyn\,(1)}\varphi \propto \frac{\partial}{\partial a}\frac{\partial}{\partial b}\,\frac{1}{\partial^{+(m+n+1)}}\left(e^{a\hat{\bar\partial}}e^{b\,\epsilon\hat{\bar q}}\partial^{+(2+m)}\varphi\,e^{-a\hat{\bar\partial}}e^{-b\,\epsilon\hat{\bar q}}\partial^{+(2+n)}\varphi\right)\Big|_{a=b=0}\,,$$

where $\hat{\bar\partial} = \frac{\bar\partial}{\partial^+}$, $\epsilon\hat{\bar q} = \epsilon^m\frac{\bar q_m}{\partial^+}$. It is not difficult to see that this form with non-negative m, n satisfies (4.19). The number of powers of ∂^+ can be determined by checking the commutation relation between two dynamical generators δ_{p^-}(Hamiltonian variation which is derived from the supersymmetry algebra) and δ_{j^-}(the boost which can also be obtained through $[\,\delta_{j^-}\,,\,\delta_{\bar q}\,]\,\varphi \;=\; \delta_s^{dyn}\varphi$), yielding that the commutator between δ_{j^-} and δ_{p^-} vanishes only when $m = n = 0$, which leads to the the same form as (4.17) written in a coherent-like form

$$\delta_s^{dyn\,(1)}\varphi = \frac{\kappa}{2}\frac{\partial}{\partial a}\frac{\partial}{\partial b}\frac{1}{\partial^+}\left[e^{a\hat{\bar\partial}}e^{b\,\epsilon\hat{\bar q}}\partial^{+2}\varphi\, e^{-a\hat{\bar\partial}}e^{-b\,\epsilon\hat{\bar q}}\partial^{+2}\varphi\right]\Big|_{a=b=0}\,.$$

It is worth noting that this is the solution that has the least number of powers of ∂^+ in the denominator, and thus the least "non-local".

The same reasoning can be applied to higher orders in κ. To order κ, we find that commutativity

$$[\,\boldsymbol{\delta}^{(-1)}\,,\,\delta_s^{dyn\,(2)}\,]\,\varphi \,+\, [\,\boldsymbol{\delta}^{(1)}\,,\,\delta_s^{dyn\,(0)}\,]\,\varphi = 0$$

requires

$$\begin{aligned}
&\boldsymbol{\delta}^{(-1)}\,\delta_s^{dyn\,(2)}\varphi \qquad\qquad (4.20)\\
&= \frac{\kappa}{4!}\Xi^{ijkl}\frac{1}{\partial^{+3}}\Bigg[-\bar d_{ijkl}\frac{\partial}{\partial^+}\varphi\,\partial^{+3}\epsilon\bar q\,\varphi + 4\,\bar d_{ijk}\partial\varphi\,\bar d_l\partial^{+2}\epsilon\bar q\,\varphi - 3\,\bar d_{ij}\partial\partial^+\varphi\,\bar d_{kl}\partial^+\epsilon\bar q\,\varphi\\
&\qquad - \bar d_{ijkl}\frac{\epsilon\bar q}{\partial^+}\,\varphi\,\partial\partial^{+3}\varphi + 4\,\bar d_{ijk}\epsilon\bar q\,\varphi\,\bar d_l\partial\partial^{+2}\,\varphi - 3\,\bar d_{ij}\partial^+\epsilon\bar q\varphi\,\bar d_{kl}\partial\partial^+\varphi\\
&\qquad + \bar d_{ijkl}\frac{\partial}{\partial^{+2}}\epsilon\bar q\,\varphi\,\partial^{+4}\varphi - 4\,\bar d_{ijk}\frac{\partial}{\partial^+}\epsilon\bar q\,\varphi\,\bar d_l\partial^{+3}\,\varphi + 3\,\bar d_{ij}\partial\epsilon\bar q\,\varphi\,\bar d_{kl}\partial^{+2}\varphi\\
&\qquad + \bar d_{ijkl}\,\varphi\,\partial\partial^{+2}\epsilon\bar q\,\varphi - 4\,\bar d_{ijk}\partial^+\,\varphi\,\bar d_l\partial\partial^+\epsilon\bar q\,\varphi + 3\,\bar d_{ij}\partial^{+2}\varphi\,\bar d_{kl}\partial\epsilon\bar q\varphi\Bigg]\,,
\end{aligned}$$

where $\epsilon\bar q$ denotes $\epsilon^m\bar q_m$, which can be written in a simpler form by rewriting it in terms of a coherent state-like form:

$$\begin{aligned}
&\boldsymbol{\delta}^{(-1)}\,\delta_s^{dyn\,(2)}\varphi \qquad\qquad (4.21)\\
&= \frac{\kappa}{2\cdot 4!}\Xi^{ijkl}\frac{\partial}{\partial a}\frac{\partial}{\partial b}\left(\frac{\partial}{\partial\eta}\right)_{ijkl}\frac{1}{\partial^{+3}}\left[\,e^{a\hat\partial}e^{b\,\epsilon\hat{\bar q}}e^{\eta\hat{\bar d}}\partial^{+4}\varphi\;e^{-a\hat\partial}e^{-b\,\epsilon\hat{\bar q}}e^{-\eta\hat{\bar d}}\partial^{+4}\varphi\right]\Bigg|_{a=b=\eta=0}\,.
\end{aligned}$$

To find $\delta_s^{dyn(2)}\varphi$ that satisfies (4.20), consider the chiral combination

$$\begin{aligned}
Z_{mnpq} &\equiv \left(\frac{\partial}{\partial\xi}\right)_{mnpq}\left(e^{\xi\hat{\bar d}}\partial^{+4}\varphi e^{-\xi\hat{\bar d}}\partial^{+4}\varphi\right)\Big|_{\xi=0}\,, \qquad (4.22)\\
&= \bar d_{mnpq}\varphi\,\partial^{+4}\varphi - 4\,\bar d_{mnp}\partial^+\varphi\,\bar d_q\partial^{+3}\varphi + 3\,\bar d_{mn}\partial^{+2}\varphi\,\bar d_{pq}\partial^{+2}\varphi\,.
\end{aligned}$$

The inhomogeneous $E_{7(7)}$ transformation of

$$Z^{ijkl} \equiv \frac{1}{4!}\epsilon^{ijklmnpq}Z_{mnpq}\,,$$

has the simple form

$$\boldsymbol{\delta}^{(-1)}Z^{ijkl} = \frac{1}{4!}\epsilon^{ijklmnpq}\,\bar d_{mnpq}\boldsymbol{\delta}^{(-1)}\varphi\,\partial^{+4}\varphi = \frac{2}{\kappa}\,\Xi^{ijkl}\,\partial^{+4}\varphi\,, \qquad (4.23)$$

which leads to the solution

$$\delta_s^{dyn\,(2)}\varphi = \frac{\kappa^2}{2\cdot 4!}\frac{\partial}{\partial a}\frac{\partial}{\partial b}\left(\frac{\partial}{\partial\eta}\right)_{ijkl}\frac{1}{\partial^{+4}}\left(e^{a\hat\partial+b\,\epsilon\hat{\bar q}+\eta\hat{\bar d}}\partial^{+5}\varphi\;e^{-a\hat\partial-b\,\epsilon\hat{\bar q}-\eta\hat{\bar d}}Z^{ijkl}\right)\Bigg|_{a=b=\eta=0}\,,$$

where we have fixed the ambiguity discussed earlier by choosing the expression with the least number of ∂^+ in the denominator. Its algebraic consistency should be checked in a future publication. This coherent state-like form is very efficient; Written out explicitly $\delta_s^{dyn\,(2)}\varphi$ consists of 60 terms.

The dynamical supersymmetry is then written in terms of the coherent state-like form,

$$\delta_s^{dyn}\,\varphi = \frac{\partial}{\partial a}\frac{\partial}{\partial b}\Bigg\{ e^{a\hat{\partial}} e^{b\,\epsilon\hat{q}}\partial^+\varphi \,+\, \frac{\kappa}{2}\frac{1}{\partial^+}\left(e^{a\hat{\bar{\partial}}+b\,\epsilon\hat{\bar{q}}}\partial^{+2}\varphi e^{-a\hat{\bar{\partial}}-b\,\epsilon\hat{\bar{q}}}\partial^{+2}\varphi\right)$$
$$+\,\frac{\kappa^2}{2\cdot 4!}\left(\frac{\partial}{\partial\eta}\right)_{ijkl}\frac{1}{\partial^{+4}}\left(e^{a\hat{\partial}+\,b\,\epsilon\hat{\bar{q}}+\eta\hat{\bar{d}}}\partial^{+5}\varphi\; e^{-a\hat{\partial}-\,b\,\epsilon\hat{\bar{q}}-\eta\hat{\bar{d}}}Z^{ijkl}\right)$$
$$+\,\mathcal{O}(\kappa^3)\Bigg\}\Bigg|_{a=b=\eta=0}\,. \tag{4.24}$$

We now use the fact, as Ananth et al [7] have shown, that the $\mathcal{N}=8$ supergravity light-cone Hamiltonian can be written as a quadratic form (to order κ^2),

$$\mathcal{H} \;=\; \frac{1}{4\sqrt{2}}\,(\mathcal{W}_m\,,\,\mathcal{W}_m)\;\equiv\;\frac{2\,i}{4\sqrt{2}}\int d^8\theta\,d^8\bar{\theta}\,d^4x\;\overline{\mathcal{W}}_m\frac{1}{\partial^{+3}}\mathcal{W}_m\,,$$

where the fermionic superfield $\mathcal{W}_m$ is the dynamical supersymmetry variation of φ

$$\delta_s^{dyn}\varphi\;\equiv\;\epsilon^m\,\mathcal{W}_m\,,$$

with

$$\mathcal{W}_m = \mathcal{W}^{(0)}_m \,+\, \mathcal{W}^{(1)}_m \,+\, \mathcal{W}^{(2)}_m + \cdots\,.$$

Up to order κ, the Hamiltonian is simply

$$\mathcal{H} = \frac{1}{4\sqrt{2}}\left[\left(\mathcal{W}^{(0)}_m\,,\,\mathcal{W}^{(0)}_m\right) + \left(\mathcal{W}^{(0)}_m\,,\,\mathcal{W}^{(1)}_m\right) + \left(\mathcal{W}^{(1)}_m\,,\,\mathcal{W}^{(0)}_m\right)\right]\,, \tag{4.25}$$

while the Hamiltonian of order κ^2 consists of three parts:

$$\mathcal{H}^{\kappa^2} = \frac{1}{4\sqrt{2}}\left[\left(\mathcal{W}^{(1)}_m\,,\,\mathcal{W}^{(1)}_m\right) + \left(\mathcal{W}^{(0)}_m\,,\,\mathcal{W}^{(2)}_m\right) + \left(\mathcal{W}^{(2)}_m\,,\,\mathcal{W}^{(0)}_m\right)\right]\,, \tag{4.26}$$

where the first part was computed by Ananth et al [7]

$$\left(\mathcal{W}^{(1)}_m,\,\mathcal{W}^{(1)}_m\right) = i\frac{\kappa^2}{2}\frac{\partial}{\partial a}\frac{\partial}{\partial b}\frac{\partial}{\partial r}\frac{\partial}{\partial s}\int d^8\theta\,d^8\bar{\theta}\,d^4x \tag{4.27}$$
$$\frac{1}{\partial^{+5}}\left(e^{a\hat{\partial}+b\hat{q}^m}\partial^{+2}\overline{\varphi}e^{-a\hat{\partial}-b\hat{q}^m}\partial^{+2}\overline{\varphi}\right)\left(e^{r\hat{\bar{\partial}}+s\hat{\bar{q}}_m}\partial^{+2}\varphi e^{-r\hat{\bar{\partial}}-s\hat{\bar{q}}_m}\partial^{+2}\varphi\right)\Big|_{a=b=r=s=0}\,,$$

and the second and third parts are complex conjugate of each other. It suffices to consider

$$\left(\mathcal{W}^{(0)}_m\,,\,\mathcal{W}^{(2)}_m\right) = i\frac{\kappa^2}{4!}\frac{\partial}{\partial a}\frac{\partial}{\partial b}\left(\frac{\partial}{\partial\eta}\right)_{ijkl}\int d^8\theta\,d^8\bar{\theta}\,d^4x \tag{4.28}$$
$$\frac{\bar{\partial}}{\partial^+}q^m\overline{\varphi}\,\frac{1}{\partial^{+7}}\left(e^{a\hat{\partial}+b\,\hat{\bar{q}}_m+\eta\hat{\bar{d}}}\partial^{+5}\varphi\; e^{-a\hat{\partial}-b\,\hat{\bar{q}}_m-\eta\hat{\bar{d}}}Z^{ijkl}\right)\Bigg|_{a=b=\eta=0}\,.$$

Integration by parts with respect to $\bar{d}$'s and use of the inside-out constraint (4.5) allow for an efficient rearrangement of terms to yield the final expression

$$\begin{aligned}&\left(\mathcal{W}^{(0)}_m\,,\,\mathcal{W}^{(2)}_m\right) \\ &\quad= -i\frac{\kappa^2}{4!}\frac{\partial}{\partial a}\frac{\partial}{\partial b}\int d^8\theta\, d^8\bar\theta\, d^4x\,\frac{\bar\partial}{\partial^{+4}}q^m d^{ijkl}\overline{\varphi}\left(e^{a\hat\partial\,+\,b\,\hat{\bar q}_m}\partial^+\overline{\varphi}\; e^{-a\hat\partial\,-\,b\hat{\bar q}_m}\frac{1}{\partial^{+4}}Z_{ijkl}\right)\Bigg|_{a=b=0}\,.\end{aligned} \tag{4.29}$$

Therefore, the Hamiltonian to order κ^2 is written as

$$\begin{aligned}\mathcal{H}^{\kappa^2} = i\,\frac{\kappa^2}{4\sqrt{2}}\int d^8\theta\, d^8\bar\theta\, d^4x\;\frac{\partial}{\partial a}\frac{\partial}{\partial b}& \\ \Bigg\{\frac{1}{2}\frac{\partial}{\partial r}\frac{\partial}{\partial s}\frac{1}{\partial^{+5}}&\left(e^{a\hat\partial+b\hat q}\partial^{+2}\overline{\varphi}e^{-a\hat\partial-b\hat q}\partial^{+2}\overline{\varphi}\right)\left(e^{r\hat{\bar\partial}+s\hat{\bar q}}\partial^{+2}\varphi e^{-r\hat{\bar\partial}-s\hat{\bar q}}\partial^{+2}\varphi\right) \\ -\;&\left[\frac{1}{4!}\frac{\bar\partial}{\partial^{+4}}q^m d^{ijkl}\overline{\varphi}\left(e^{a\hat\partial\,+\,b\,\hat{\bar q}_m}\partial^+\overline{\varphi}\; e^{-\,a\,\hat\partial\,-\,b\,\hat{\bar q}_m}\frac{1}{\partial^{+4}}Z_{ijkl}\right)+c.c.\right]\Bigg\}\Bigg|_{a=b=r=s=0}\,,\end{aligned} \tag{4.30}$$

to be compared with the 96 terms of Ananth et al [7]!

5. Conclusions and outlook

In this paper, we have explicitly derived the non-linear $E_{7(7)}$ transformations on the 256 physical fields of $\mathcal{N}=8$ Supergravity. We found that they can be elegantly written in terms of the constrained chiral superfield in light-cone superspace, at least to order κ. In this gauge, all fields, including the graviton transform under $E_{7(7)}$. This is to be compared with the original covariant formulation of Cremmer and Julia in which the graviton is invariant. The process of gauge fixing, elimination of the unphysical degrees of freedom, and the preservation of kinematical supersymmetry, requires that all fields transform. We note that the form of the Lagrangian in the LC_2 formulation that involves no time derivatives in the interaction is unique. Since the kinetic term contains the time derivatives, any non-linear field redefinition would reintroduce time derivatives in the interaction terms. Hence there is no possibility to find a change of field degrees of freedom which could lead to a graviton field invariant under $E_{7(7)}$. The simplicity of our result lends hope for the existence of a compact all-orders in κ formulation of these non-linear $E_{7(7)}$ transformations[1] (as in non-linear σ-models).

In light-cone superspace, the symmetries of the theory can be identified with the semi-direct product of the $\mathcal{N}=8$ SuperPoincaré group with $E_{7(7)}$. Its non-linearity indicates that while the eight supercharges do transform under its compact subgroup, they are invariant under the $E_{7(7)}/\,\mathrm{SU}(8)$ coset. These symmetries can now be used to find the dynamical

[1]As this paper was being revised, a preprint appeared [20] which provided the action of the $E_{7(7)}$ generators to all orders in κ. We note that in their analysis, performed in the covariant formalism, only the vectors and scalars transform under the coset $E_{7(7)}/\,\mathrm{SU}(8)$, but in the LC_2 formalism we use all physical fields transform. Therefore the all-order expressions for the coset transformation on fields are different from those in the covariant formalism.

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supersymmetry, and then the light-cone Hamiltonian. We use *both* $E_{7(7)}$ and the super-Poincaré algebra to find the dynamical supersymmetry transformations of the superfield to order κ^2. In the superfield language, it is also a remarkably simple expression, which suggests that an all-order in κ expression may be feasible. The light-cone Hamiltonian that follows is equally simple.

Our results indicate that the light-cone superspace formalism, although awkward for many detailed calculations, produces tractable results that are bound to shed further light on the structure of $\mathcal{N}=8$ Supergravity.

The formalism is also suitable to examine how general these symmetries are, such as if they can be extended to other dimensions. All these questions will be discussed and more detailed proofs of the formulae in this paper in forthcoming papers [11].

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